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A model of the undular bore on a viscous fluid

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A solution for the weak bore is found in which the mean profile is dominated by viscosity, so that the velocity variation is given essentially by a quasi-uniform Poiseuille flow. It is found that such a transition between flows of different depths is possible provided the Froude number is less than 1·58. The possibility of superposing an inviscid perturbation on such a flow is then investigated. Under favourable circumstances the effect of this perturbation is to add to the profile of the free surface a term which decays exponentially in front of the bore, but is oscillatory behind it.

1. Introduction

The classical theory of the bore (Lamb 1932) is based on a model consisting of two uniform streams connected by a transition through which mass and momentum are conserved. For a given upstream velocity and given strength this is sufficient to determine the velocity of propagation of the bore and, further, predict a loss of energy which is assumed to be dissipated in turbulence.

This model is generally accepted for a bore of substantial strength, but experiment shows that weak bores have a stationary train of waves behind them and exhibit no tendency to break (Favre 1935). Lemoine (1948) suggests that in these circumstances energy is lost by radiation through the wave train rather than by turbulence.

A more detailed account is given by Benjamin & Lighthill (1954), who also suggest that perhaps the amplitude of the waves is too large for a linear theory to be applicable, and proceed to develop a theory based on a cnoidal wave train. Their conclusion is that a wave train is possible provided that some energy is dissipated by friction at the bore. This need not be the full amount required by the classical theory, but if there is no such frictional dissipation only the solitary wave with infinite wavelength is possible.

There is no doubt that some frictional dissipation will occur in practice, but the assumption that this takes place at the bore, through the mechanism of turbulence, seems to be worth further investigation. For a weak bore it may be sufficient to invoke the effect of laminar friction at the bed. Such a mechanism would act systematically over the whole extent of the stream, rather than locally in the region of transition. The fact that experiment indicates that the weak bore is smooth with no tendency to break lends some support to this possibility. Sturtevant (1965), in his discussion of the experiments of Favre (1935), also comes to the same conclusion.

The problem studied in this paper is developed from this point of view. It is convenient to define co-ordinate axes (Ox, Oy) relative to which the transition is steady and the flow at infinity is along the x-axis (figure 1). To obtain a steady solution is it is necessary to include a body force in the direction of flow to counteract the effect of the viscous stresses. The flow is therefore assumed to take place along a slightly inclined bed, so that the gravitational acceleration has components $(-g_1, -g_2)$, with $g_1 \ll g_2$.

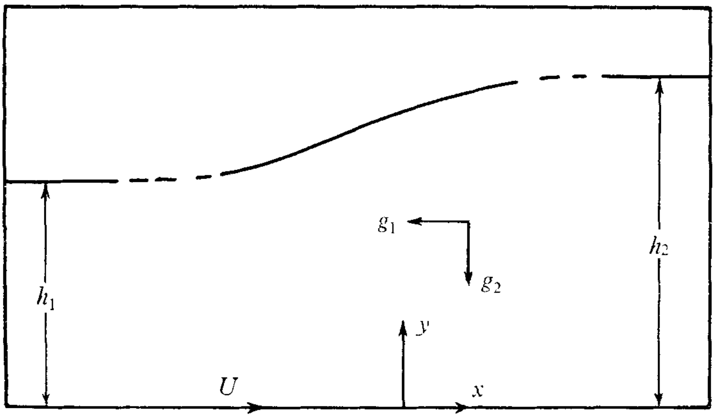

FIGURE 1.

We first note that if, relative to the chosen set of axes, the x-component of velocity, u, is equal to U at the bed ($y = 0$) then a uniform flow is possible with constant depth h, where

$$u = U - \frac{g_1}{\nu}(hy - \tfrac{1}{2}y^2) \tag{1.1}$$

and ν is the kinematic viscosity. This is the usual solution for Poiseuille flow with the boundary conditions

$$\left.\begin{aligned} u &= U \quad \text{at} \quad y = 0, \\ \partial u/\partial y &= 0 \quad \text{at} \quad y = h, \end{aligned}\right\} \tag{1.2}$$

and the y-component of velocity identically zero. Thus if there is to be a transition from such a uniform flow with depth h_1 to another uniform flow with depth h_2 ($> h_1$), constancy of mass flux requires that

$$\int_0^{h_1} u(h_1, y)\, dy = \int_0^{h_2} u(h_2, y)\, dy, \tag{1.3}$$

or

$$h_1^2 + h_1 h_2 + h_2^2 = 3\nu U/g_1. \tag{1.4}$$

This is a necessary condition for the type of transition considered in the present problem. It can be regarded as determining the velocity of propagation of the transition when the upstream and downstream depths are known.

For the subsequent analysis it will be useful to define the length parameter h_0, where

$$h_0^2 = \tfrac{1}{3}(h_1^2 + h_1 h_2 + h_2^2) = \left(\frac{h_1 + h_2}{2}\right)^2 + \frac{1}{3}\left(\frac{h_2 - h_1}{2}\right)^2. \tag{1.5}$$

For a weak bore, h_0 can be regarded as the mean height. In terms of h_0, condition (1.4) is simply

$$\nu U/g_1 h_0^2 = 1. \tag{1.6}$$

2. The basic equations

For steady, two-dimensional, incompressible, viscous flow with velocity (u, v), pressure p and density ρ, the governing equations are

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}-g_1+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}-g_2+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0. \end{aligned}\right\} \tag{2.1}$$

In terms of the non-dimensional variables

$$u' = \frac{u}{U}, \quad v' = \frac{v}{\alpha U}, \quad p' = \frac{p}{\rho g_2 h_0}, \quad x' = \frac{\alpha x}{h_0}, \quad y' = \frac{y}{h_0}, \tag{2.2}$$

where α is the maximum slope of the free surface, the above equations become

$$\left.\begin{aligned} F^2\left(u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}\right) &= -\frac{\partial p'}{\partial x'}-g+g\left(\alpha^2\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}\right), \\ \alpha^2F^2\left(u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}\right) &= -\frac{\partial p'}{\partial y'}-1+\alpha^2 g\left(\alpha^2\frac{\partial^2 v'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}\right), \\ \frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'} &= 0, \end{aligned}\right\} \tag{2.3}$$

where

$$F^2 = U^2/g_2 h_0, \quad g = g_1/\alpha g_2 \tag{2.4}$$

and equation (1.6) has been used.

It is possible to simplify equation (2.3) in two ways, both of which are relevant in the present problem. If it be assumed that $\alpha \ll 1$ and that $g = O(1)$, then the equations approximate to

$$\left.\begin{aligned} F^2\left(u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}\right) &= -\frac{\partial p'}{\partial x'}-g+g\frac{\partial^2 u'}{\partial y'^2}, \\ 0 &= -\frac{\partial p'}{\partial y'}-1, \\ \frac{\partial u'}{\partial x'}+\frac{\partial u'}{\partial y'} &= 0. \end{aligned}\right\} \tag{2.5}$$

It will be assumed that the average profile, dominated by viscosity, is determined by (2.5).

On the other hand, if α, though still fairly small, is large enough to make $g \ll 1$, then the appropriate approximate form is

$$\left.\begin{aligned} F^2\left(u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}\right) &= -\frac{\partial p'}{\partial x'}, \\ \alpha^2F^2\left(u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}\right) &= -\frac{\partial p'}{\partial y'}-1, \\ \frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'} &= 0. \end{aligned}\right\} \tag{2.6}$$

The last equations will be used to study the effect of a perturbation of the average profile. This perturbation takes place on a scale small enough for it to be independent of the influence of viscosity.

3. The mean profile

We first look for a transition between two flows at constant depths h_1 and h_2 satisfying (1.4), which is consistent with the assumptions made in deriving equations (2.5). These equations, written now in dimensional form, are

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}-g_1+\nu\frac{\partial^2 u}{\partial y^2}, \\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial y}-g_2, \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0. \end{aligned}\right\} \tag{3.1}$$

The last of these equations implies the existence of a stream function, which is written in the form

$$\Psi = U\left\{y-\frac{h_m y^2}{2h_0^2}+\frac{1}{6}\frac{y^3}{h_0^2}\right\}+\psi, \tag{3.2}$$

where the depth h_m is a slowly varying function of x, and ψ is a small correction to the quasi-uniform Poiseuille flow. The significant contribution to $\partial\Psi/\partial y$ is of the form (1.1) when the expression for h_0^2 obtainable from (1.6) is used.

The second of equations (3.1) gives

$$(p-p_0)/\rho = g_2(h_m-y), \tag{3.3}$$

where p_0 is the atmospheric pressure. The first equation then becomes, with the help of (3.2) and (3.3),

$$\nu\psi_{yyy} = g_2 h_m' + \{\Psi_y\Psi_{xy}-\Psi_x\Psi_{yy}\}. \tag{3.4}$$

If the contribution from ψ to the right-hand side of (3.4) is omitted, this equation becomes

$$\psi_{yyy} = \frac{g_2 h_m'}{\nu}\left\{1-\frac{F^2 y}{h_0}+\tfrac{1}{2}F^2 h_m y^2/h_0^3\right\}. \tag{3.5}$$

The conditions under which (3.4) can be approximated by (3.5) will be discussed after the solution is obtained. For the moment it is assumed that ψ can be calculated from (3.5). Then, together with the boundary conditions

$$\left.\begin{aligned} \psi = \psi_y = 0 \quad &\text{at} \quad y = 0, \\ \psi_{yy} = 0 \quad &\text{at} \quad y = h_m, \end{aligned}\right\} \tag{3.6}$$

equation (3.5) can be integrated to give

$$\psi = \frac{g_2 h'_m}{\nu}\left\{\tfrac{1}{6}y^3 - \tfrac{1}{2}h_m y^2 - \frac{F^2}{24h_0}(y^4 - 6h_m^2 y^2) + \frac{F^2 h_m}{120 h_0^3}(y^5 - 10h_m^3 y^2)\right\}. \tag{3.7}$$

Finally, from conservation of mass,

$$\begin{aligned} (\Psi)_{y=h_m} &= \text{const.} \\ &= U\left\{h_1 - \frac{h_1^3}{2h_0^2} + \frac{1}{6}\frac{h_1^3}{h_0^2}\right\} \end{aligned} \tag{3.8}$$

and hence, by substitution from (3.2) and (3.7),

$$h_m^{-3}(h_m - h_1)(h_2 - h_m)(h_1 + h_2 + h_m) = g_1^{-1} g_2 h'_m\left(1 - \tfrac{5}{8}F^2\frac{h_m}{h_0} + \tfrac{9}{40}F^2\frac{h_m^3}{h_0^3}\right). \tag{3.9}$$

It follows immediately from (3.9) that h_m/h_0 is a function of $g_1 x/g_2 h_0$, which shows the slowly varying character of the profile. A solution which varies gradually and monotonically from h_1 at $x = -\infty$ to h_2 at $x = +\infty$ is possible provided that

$$1 - \tfrac{5}{8}F^2\frac{h_m}{h_0} + \tfrac{9}{40}F^2\frac{h_m^3}{h_0^3} > 0 \tag{3.10}$$

for all h_m in the range $h_1 \leqslant h_m \leqslant h_2$, and does not become too small in this range. The inequality is satisfied for all h_m within the prescribed range if

$$F^2 < 3^{\frac{1}{2}}\, 36/25 = 2{\cdot}494, \tag{3.11}$$

except for extremely weak bores where

$$h_1/h_0 > (25/27)^{\frac{1}{2}} \quad \text{or} \quad (h_2 - h_1)/h_0 < 0{\cdot}04. \tag{3.12}$$

The required condition on F is then

$$F^2 < \frac{5}{2}\left\{1 - \frac{(h_2 - h_1)}{16h_0} + O\left(\frac{h_2 - h_1}{h_0}\right)^2\right\}. \tag{3.13}$$

Thus for all practical purposes, the condition on F is

$$F^2 < 2{\cdot}5 \quad \text{or} \quad F < 1{\cdot}58. \tag{3.14}$$

All these properties of the profile are clearly brought out in the simplified solution of (3.9), based on the assumption that $(h_2 - h_1)/h_0 \ll 1$. We may then write

$$3h_0^{-2}(h_m - h_1)(h_2 - h_m) = g_1^{-1} g_2 h'_m (1 - 2F^2/5), \tag{3.15}$$

or

$$h_m = h_0 + \tfrac{1}{2}(h_2 - h_1)\tanh(\beta x/h_0), \tag{3.16}$$

where

$$\beta = \frac{3g_1(h_2 - h_1)}{2g_2(1 - 2F^2/5)\,h_0}, \tag{3.17}$$

and the slowly varying character of the profile is assured provided that $\beta \ll 1$.

It is now a simple matter to discuss the validity of (3.5) as an approximation to (3.4). From (3.5) ψ is of order $g_2 h_0^3 h_m'/\nu$ and the ratio of the terms omitted to the terms retained is in order of magnitude $g_2 h_0 h_m''/g_1 h_m'$ or $(h_2 - h_1)/h_0$. The approximation is therefore valid for a weak transition, the same assumption used to derive (3.16).

We note in passing that equation (3.15) also has solutions appropriate to a steady bore of negative strength ($h_2 < h_1$), the conditions for which are that the change in height should not be large and that $F^2 > 2.5$. This is in contrast with inviscid theory, where such a bore is necessarily unsteady. It would be interesting to see if these solutions can be reproduced experimentally.

4. The inviscid contribution to the profile

We now look for a further contribution to the profile in which the length scale, though sufficiently large to ensure that the slope of the free surface remains small, is short compared with that which determines the mean profile. Accordingly, we use equations (2.6) which, in dimensional form, are

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x},\\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}-g_2,\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0. \end{aligned}\right\} \tag{4.1}$$

Let the stream function be $\Psi+\phi$, where ϕ represents a perturbation of the mean flow and the dependence of Ψ on x can now be ignored, at least locally. Equations (4.1) become

$$\left.\begin{aligned} (W+\phi_y)\,\phi_{xy}-\phi_x(W+\phi_{yy}) &= -\frac{1}{\rho}\frac{\partial p}{\partial x},\\ -(W+\phi_y)\,\phi_{xx}+\phi_x\phi_{xy} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}-g_2, \end{aligned}\right\} \tag{4.2}$$

where $W(y) = \Psi_y$.

The second of equations (4.2) gives, on integration,

$$\frac{p}{\rho} = g_2(h_m+\eta-y)-\int_y^{h_m+\eta}\{(W+\phi_y)\,\phi_{xx}-\phi_x\phi_{xy}\}\,dy, \tag{4.3}$$

where the free surface is defined by $y = h_m+\eta$. The first equation then gives

$$(W+\phi_y)\,\phi_{xy}-\phi_x(W+\phi_{yy}) = -g_2\eta_x+\frac{\partial}{\partial x}\int_y^{h_m+\eta}\{(W+\phi_y)\,\phi_{xx}-\phi_x\phi_{xy}\}\,dy. \tag{4.4}$$

To solve equation (4.4) we anticipate that the appropriate solution is a wave of low frequency for which the x derivatives are in decreasing order of magnitude. If the amplitude is also small, a first approximation is then given by

$$\frac{\partial}{\partial y}\frac{\phi_{1x}}{W} = -\frac{g_2\eta_x}{W^2}, \tag{4.5}$$

or

$$\phi_1 = -g_2\eta W I \tag{4.6}$$

where
$$I(y) = \int_0^y \frac{dX}{W^2(X)}. \tag{4.7}$$

On a linear theory, the next approximation is obtained by the inclusion of the remaining linear term on the right-hand side of (4.4). A further improvement, in the spirit of cnoidal wave theory, is achieved if terms which make a contribution to ϕ of order η^2 are also included. The basis of this approximation is described by Benjamin (1962), who obtains equations similar to those derived in this section but by a different approach. Briefly, in a cnoidal wave of small amplitude a, the orders of magnitude are

$$\eta = O(a), \quad \eta_x = O(a^{\frac{3}{2}}), \quad \eta_{xx} = O(a^2)$$

and the theory is a second-order approximation in a.

Most of the conclusions of this investigation will be based on the simpler linear theory. This is, of course, more severe than cnoidal-wave theory in its restrictions on the amplitude, but probably illustrates the main qualitative features of the flow. It will, however, be convenient to have on display the basic equation for the more accurate theory.

Thus the next approximation to ϕ is calculated from

$$\frac{\partial}{\partial y}\frac{\phi_x}{W+\phi_{1y}} = -\frac{g_2\eta_x}{(W+\phi_{1y})^2} - \frac{g_2}{W^2}\frac{\partial}{\partial x}\int_y^{h_m} \eta_{xx} W^2 I\, dy, \tag{4.8}$$

which gives, to the required order of accuracy,

$$\phi_x = -g_2 WI\eta_x + g_2^2(W'I^2 + 2I/W - 3JW)\eta\eta_x - g_2 WK\eta_{xxx}, \tag{4.9}$$

$$\phi = -g_2 WI\eta + \tfrac{1}{2}g_2^2(W'I^2 + 2I/W - 3JW)\eta^2 - g_2 WK\eta_{xx}, \tag{4.10}$$

where
$$J(y) = \int_0^y \frac{dX}{W^4(X)}, \tag{4.11}$$

$$K(y) = \int_0^y dX \int_X^{h_m} \frac{I(Y)\, W^2(Y)}{W^2(X)}\, dY. \tag{4.12}$$

The equation satisfied by η is found from conservation of mass. After some manipulation one finds that

$$\begin{aligned}\Psi(h_m+\eta) + \phi(h_m+\eta) &= \Psi(h_m) + W(h_m)\,[1-g_2 I(h_m)]\left[\eta + \eta^2\left\{\frac{W'(h_m)}{2W(h_m)} - g_2\frac{I(h_m)\,W'(h_m)}{W^2(h_m)} - \frac{g_2}{W^2(h_m)}\right\}\right] \\ &\quad - \tfrac{3}{2}g_2^2 W(h_m)\,J(h_m)\,\eta^2 - g_2 W(h_m)\,K(h_m)\,\eta_{xx}\end{aligned} \tag{4.13}$$

and the boundary condition at the free surface then gives

$$\begin{aligned}0 = [1-g_2 I(h_m)]\left[\eta + \eta^2\left\{\frac{W'(h_m)}{2W(h_m)} - g_2\frac{I(h_m)\,W'(h_m)}{W^2(h_m)} - \frac{g_2}{W^2(h_m)}\right\}\right] \\ - \tfrac{3}{2}g_2^2 J(h_m)\,\eta^2 - g_2 K(h_m)\,\eta_{xx}.\end{aligned} \tag{4.14}$$

The simplified equation,

$$0 = \{1-g_2 I(h_m)\}\,\eta - \tfrac{3}{2}g_2^2 J(h_m)\,\eta^2 - g_2 K(h_m)\,\eta_{xx}, \tag{4.15}$$

is that used by Benjamin (1962) to study the solitary wave. The omitted term is clearly uniformly small compared with those retained, but the second-order term can be comparable with the linear term if $(1-g_2 I)\, Jg_2^{-2}$ is comparable with the amplitude of η. If, however, η is sufficiently small the linearized equation will suffice. Then

$$0 = \{1-g_2 I(h_m)\}\,\eta - g_2 K(h_m)\,\eta_{xx}. \tag{4.16}$$

Since $K(h_m)$ is clearly positive, it follows that the perturbation is oscillatory if $g_2 I > 1$ and exponential if $g_2 I < 1$. Now on a scale of length used for the mean profile, $I(h_m)$ does in fact vary slowly with x through its dependence on h_m, and the interesting case, as far as the weak bore is concerned, is that for which $(1-g_2 I)$ is positive at $x = -\infty$ and negative at $x = +\infty$. When this is so the equation for η takes the form

$$\eta_{xx} + \lambda\eta = 0, \tag{4.17}$$

where

$$\lambda = K^{-1}(h_m)\,\{I(h_m) - g_2^{-1}\} \tag{4.18}$$

and is a slowly varying monotonic function of x which increases from a small negative value at $x = -\infty$ to a small positive value at $x = +\infty$. There are various techniques for dealing with such an equation. A simple method of solution is as follows†. Let

$$\eta = aA(b), \tag{4.19}$$

where a and b are functions of x. Substitution in (4.17) gives

$$ab_x^2 A'' + (ab_{xx} + 2a_x b_x)\,A' + (a_{xx} + \lambda a)\,A = 0. \tag{4.20}$$

If a is chosen so that

$$ab_{xx} + 2a_x b_x = 0,$$

or

$$a = b_x^{-\frac{1}{2}}, \tag{4.21}$$

and A satisfies

$$A'' + bA = 0, \tag{4.22}$$

the equation which determines b is

$$bb_x^2 - \tfrac{3}{4} b_x^{-2} b_{xx}^2 + \tfrac{1}{2} b_x^{-1} b_{xxx} = \lambda. \tag{4.23}$$

Since λ is a slowly varying function of x, the derivatives of b are in decreasing order of magnitude and this equation may be approximated by

$$bb_x^2 = \lambda. \tag{4.24}$$

If $\lambda = 0$ when $x = c$, an appropriate solution for b is given by

$$\left.\begin{aligned} \tfrac{2}{3} b^{\frac{3}{2}} &= \int_c^x \lambda^{\frac{1}{2}}\,dx \qquad (x \geqslant c),\\ \tfrac{2}{3}(-b)^{\frac{3}{2}} &= \int_x^c (-\lambda)^{\frac{1}{2}}\,dx \quad (x \leqslant c).\end{aligned}\right\} \tag{4.25}$$

The variable b is then a monotonic increasing function of x which is zero when $x = c$.

† A more careful argument is given by Erdelyi (1956).

Equation (4.22) for A is the Airey equation (Jeffries & Jeffries 1956). The appropriate solution, which is everywhere bounded, is such that

$$A(0) = \frac{1}{3^{\frac{1}{6}}2\pi}(-\tfrac{2}{3})!, \tag{4.26}$$

$$A(b) \sim \frac{1}{2\pi^{\frac{1}{2}}(-b)^{\frac{1}{4}}}\exp\{-\tfrac{2}{3}(-b)^{\frac{3}{2}}\} \quad (b \to -\infty) \tag{4.27}$$

$$\sim \frac{1}{\pi^{\frac{1}{2}}b^{\frac{1}{4}}}\sin\{\tfrac{2}{3}b^{\frac{3}{2}}+\tfrac{1}{4}\pi\} \qquad (b \to \infty). \tag{4.28}$$

It follows, from (4.19) and the subsequent relations, that

$$\frac{\eta}{\eta_0} \sim \frac{3^{\frac{1}{6}}\pi^{\frac{1}{2}}}{(-\frac{2}{3})!(-\lambda^{\frac{1}{4}})}\exp\left\{-\int_x^c(-\lambda)^{\frac{1}{2}}\,dx\right\} \quad (x \to -\infty) \tag{4.29}$$

$$\sim \frac{3^{\frac{1}{6}}2\pi^{\frac{1}{2}}}{(-\frac{2}{3})!\,\lambda^{\frac{1}{4}}}\sin\left\{\int_c^x \lambda^{\frac{1}{2}}\,dx+\tfrac{1}{4}\pi\right\} \quad (x \to +\infty), \tag{4.30}$$

where η_0 is the value of η at $\lambda = 0$ $(x = c)$.

The asymptotic behaviour shows clearly the transition to oscillatory flow as $x \to \infty$, and it remains to discuss in more detail the conditions for which such a solution is possible. It is not difficult to verify that $\{g_2 I(h_m)-1\}$ is an increasing function of h_m so that the type of profile discussed above can occur if

$$g_2 I(h_m) = g_2\int_0^{h_m}\frac{dX}{W^2(X)} = 1 \tag{4.31}$$

for some value of h_m in the mean flow. To evaluate $I(h_m)$ it is sufficient to take

$$W(y) = U\{1-h_m y/h_0^2+\tfrac{1}{2}(y^2/h_0^2)\}, \tag{4.32}$$

then
$$g_2 I(h_m) = \frac{g_2}{U^2}\int_0^{h_m}\frac{dX}{\left\{1-\frac{h_m X}{h_0^2}+\frac{1}{2}\frac{X^2}{h_0^2}\right\}^2}$$

$$= \frac{2}{F^2(2-h_m^2/h_0^2)^{\frac{3}{2}}}\left\{\sin^{-1}\left(\frac{h_m}{2^{\frac{1}{2}}h_0}\right)+\frac{h_m}{2h_0}(2-h_m^2/h_0^2)^{\frac{1}{2}}\right\}. \tag{4.33}$$

Thus equation (4.31) is equivalent to a relation between F and the critical value of h_m/h_0. This relation is represented by the lower curve of figure 2. For a given value of F the critical value of h_m/h_0 may be regarded as the upper bound of h_1/h_0 if the profile ahead of the bore is to decrease exponentially. The corresponding lower bound of h_2/h_0 is shown in the upper curve of figure 2, which is calculated from (1.5). If, for example, $F = 1{\cdot}4$, then the transition to oscillatory flow will occur when $h_m/h_0 = 0{\cdot}925$. This is possible if $h_1/h_0 < 0{\cdot}925$ which is equivalent to

$$h_2/h_0 > 1{\cdot}075 \quad \text{or} \quad (h_2-h_1)/h_0 > 0{\cdot}150. \tag{4.34}$$

One interesting feature of the results is that the limiting value of F $(= 1{\cdot}60)$, for which a transition to oscillatory flow is nominally possible in an infinitesimally weak bore, is extremely close to the limiting value of F $(= 1{\cdot}58)$ for which a

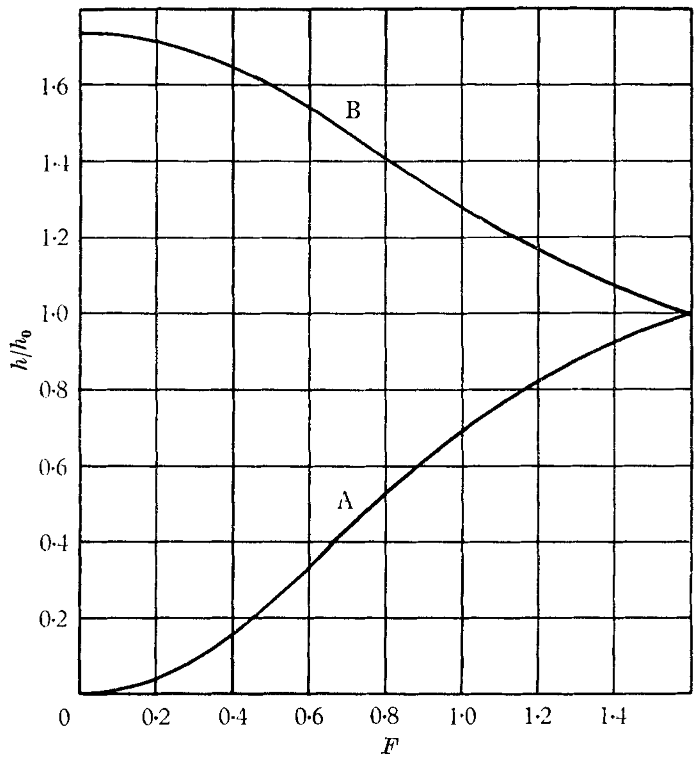

FIGURE 2. A, Critical value of h_m/h_0 for transition to oscillatory flow; B, minimum value of h_2/h_0 for transition to occur.

monotonically increasing profile for the mean flow is possible (see (3.14)). Whether or not this is fortuitous is an open question, but the two conditions augment each other to suggest that the change in mean level should not be too small if this type of bore is to be possible.

If Benjamin & Lighthill (1954) are correct in their assertion that the amplitude of the oscillations behind the bore is too large to be described by linear theory, then the present analysis would have to be considerably refined. Strictly speaking the procedure adopted here is not in fact applicable, for on a non-linear theory there is a coupling between the viscous mean profile and the inviscid cnoidal profile, and the two calculations cannot proceed independently. The qualitative features, however, seem clear. The exponential behaviour well ahead of the bore will remain, since this can certainly be treated on a linear theory sufficiently far from the bore. Behind the bore, the sinusoidal character of the inviscid contribution to the profile will give way to a cnoidal wave governed by an equation similar to (4.15), but in which the mean profile is not known *a priori*. The correct approach is undoubtedly along the lines of Whitham's theory of non-linear dispersive waves (Whitham 1964), but this must await further investigation.

One final comment is appropriate on the orders of magnitude involved in a typical situation. If in c.g.s. units we take

$$h_0 = O(1), \quad g_2 = O(10^3), \quad \nu = O(10^{-2}),$$

then equations (1.6), (2.4) and (3.11) combine to give

$$g_1/g_2 = O(10^{-3}).$$

Thus, unless h_0 is considerably smaller than $O(1)$, only a very gentle slope is required to achieve this type of bore in a fluid such as water. Experimentally it is commonly achieved from a bore, created in a tank with a level bed, which has become weak enough to display the undular profile. When viscous dissipation is operative this is a time-dependent problem with the fluid far ahead of the bore presumably quiescent. For obvious theoretical reasons the conditions in the present analysis are chosen so that a steady profile can be studied. From the point of view of comparison the unsteadiness will not be too important, if as is probably the case, large time intervals are required to produce observable changes in the profile. Whether or not the change in the boundary condition ahead of the bore produces markedly different results remains to be seen. The present analysis does however produce a solution with a profile similar to that observed experimentally, and perhaps brings out the mechanism by which this is achieved.

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